# Appendix E
# Discrete Time Markov Chains

This material is drawn from the following website for the University of Texas at Dallas, which provides Fall 2003 course notes for "Computer Sciences 6352: Performance of Computer Systems and Networks" (instructor: Assistant Professor Jason Jue): http://www.utdallas.edu/~jjue/cs6352/markov.

## Discrete-Time Markov Chains

A Markov chain is a discrete state space process in which the next state depends only on the present state.

For a discrete time system, if $X_n$ is the state of the system at time $n$, then $\{X_n : n \geq 0\}$ is a Markov chain if:

$$Pr\left[X_n = j | X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \ldots, X_0 = i_0\right] = Pr\left[X_n = j | X_{n-1} = i_{n-1}\right],$$

i.e., the state (*j*) of the system at time $n$ depends only on the state (*i*) of the system at time $n-1$, and does not depend on any other state before time $n-1$.

## State Probabilities

The state probability, denoted as $\pi_j(n)$, is the probability that the process is in state $j$ at time $n$.

$$\pi_j(n) = Pr\{X_n = j\}$$

The state probability vector is denoted as $\Pi(n)$, and consists of all of the state probabilities for a given time $n$.

$$\Pi(n) = \left[\begin{array}{cccc} \pi_0(n) & \pi_1(n) & \pi_2(n) & \cdots \end{array}\right]$$

Note that the sum over the elements in $\Pi(n)$ is equal to 1.

$$\sum_j \pi_j(n) = 1$$

## Transition Probabilities

The *one-step transition probability* is the probability of transitioning from one state to another in a single step. The Markov chain is said to be time homogeneous if the transition probabilities from one state to another are independent of time index $n$.

$$p_{ij} = Pr\{X_n = j | X_{n-1} = i\}$$

The *transition probability matrix*, $P$, is the matrix consisting of the one-step transition probabilities, $p_{ij}$.

The $m$-*step transition probability* is the probability of transitioning from state $i$ to state $j$ in $m$ steps.

$$p_{ij}^{(m)} = Pr\{X_{n+m} = j | X_n = i\}$$

The $m$-step transition matrix whose elements are the $m$-step transition probabilities $p_{ij}^{(m)}$ is denoted as $P^{(m)}$.

The $m$-step transition probabilities can be found from the single-step transition probabilities as follows.

To transition from $i$ to $j$ in $m$ steps, the process can first transition from $i$ to $r$ in $m-k$ steps, and then transition from $r$ to $j$ in $k$ steps, where $0 < k < m$.

$$p_{ij}^{(m)} = \sum_r p_{ir}^{m-k} p_{rj}^{k}$$

In matrix form, this becomes:

$$P^{(m)} = P^{(m-k)} P^{(k)}$$

Setting $k = m - 1$ yields:

$$P^{(m)} = P \cdot P^{(m-1)}$$

From this equation we can see that:

$$P^{(m-1)} = P \cdot P^{(m-2)}$$

Substituting this back into the previous equation yields:

$$P^{(m)} = P \cdot P \cdot P^{(m-2)}$$

Continuing these substitutions, eventually we have:

$$P^{(m)} = P \cdot P \cdot P \cdots P = P^m$$

Therefore, the $m$-step transition probability matrix can be found by multiplying the single-step probability matrix by itself $m$ times.

The state vector at time $m$ can also be found in terms of the transition probability matrix and the intial state vector $\Pi(0)$. We first observe that:

$$\pi_j(m) = \sum_i \pi_i(m-1)p_{ij}$$

In vector and matrix form, this becomes:

$$\Pi(m) = \Pi(m-1)P$$

We also find that, through substitution:

$$\Pi(m-1) = \Pi(m-2)P$$

or,

$$\Pi(m) = \Pi(m-2)P \cdot P$$

Continuing the substitution yields:

$$\Pi(m) = \Pi(0)P^m$$

where $\Pi(0)$ is the vector containing the initial probabilities of being in each state at time 0.

## Long-Run Behavior of Markov Chains

As the time index $m$ approaches infinity, a Markov chain may settle down and exhibit steady-state behavior. If the following limit exists:

$$\lim_{m \to \infty} p_{ij}^{(m)} = \pi_j$$

for all values of $i$, then the $\{\pi_j\}$ are the limiting or steady-state probabilities.

Looking at the state probability as $m$ approaches infinity, we see that:

$$\begin{aligned}
\lim_{m \to \infty} \pi_j(m) &= \lim_{m \to \infty} \sum_i \pi_i(0) p_{ij}(m) \\
&= \sum_i \pi_i(0) \lim_{m \to \infty} p_{ij}(m) \\
&= \sum_i \pi_i(0) \pi_j \\
&= \pi_j \sum_i \pi_i(0) \\
&= \pi_j
\end{aligned} \tag{1}$$

When the limiting probabilities exist, they can be found using the following equations:

$$\Pi = \Pi P$$

and

$$\sum_i \pi_i = 1$$

where

$$\Pi = \left[ \begin{array}{cccc} \pi_0 & \pi_1 & \pi_2 & \cdots \end{array} \right]$$